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ON THE METHOD OF HOMOGENEOUS SOLUTIONS IN MIXED PROBLEMS OF THE THEORY OF ELASTICITY FOR A TRUNCATED WEDGE AND A RING SECTOR*

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Two proper mixed problems of pure shear by a band stamp along the generatrix of a cylindrical elastic body are considered. The body cross-section occupies a region bounded by the sides of a wedge and by two concentric circles with the centre at the wedge apex. In the first problem the stamp is fixed to the plane face of the body, the other plane face is also fixed, while the cylindrical faces are either fixed for free of stresses. In the second problem the stamp is fixed on the outer cylindrical surface, with the inner cylindrical surface fixed, and the two plane faces either fixed or free of stresses.

The method of homogeneous solutions /1/ is used to solve the problem. The method enables the problem to be reduced to an investigation of an infinite set of linear algebraic equations of the second kind of high quality with exponentially decreasing matrix elements and right-hand sides. Such systems are normal Poincaré-Koch systems(see, for instance, /2/), and their solution can be obtained using the method of reduction for any values of the problem parameters.

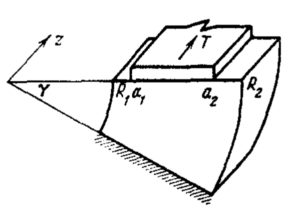

Fig. 1

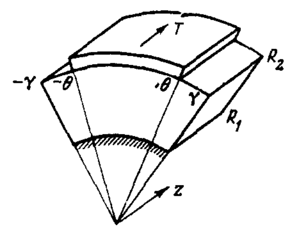

Fig. 2

1. **Shear by a stamp of a truncated wedge.** We consider, in cylindrical coordinates (r, φ, z), an elastic body occupying the region $R_1 \leqslant r \leqslant R_2$, $0 \leqslant \varphi \leqslant \gamma$, $-\infty < z < \infty$, which we shall call a truncated wedge, since the investigation of this problem is based on the solution of some problems for an infinite wedge.

Problem 1 on the shear by a stamp of a truncated wedge is equivalent to the following boundary value problem in the displacement function $w(r, z)$ along the z axis:

$$\Delta w = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{1.1}$$

$$\begin{aligned} & w = \delta \ (\varphi = \gamma,\ a_1 \leqslant r \leqslant a_2),\ w = 0\ (\varphi = 0,\ R_1 \leqslant r \leqslant R_2) \\ & \tau_{\varphi z} = \frac{G}{r}\frac{\partial w}{\partial \varphi} = 0 \quad (\varphi = \gamma,\ a_1 < r < R_1,\ a_2 < r < R_2) \\ & w = 0\ (r = R_1, r = R_2,\ 0 \leqslant \varphi \leqslant \gamma) \end{aligned} \tag{1.2}$$

where δ is the displacement of the stamp, G is the shear modulus, and $\tau_{\varphi z}$ are the tangential stresses. The last condition implies that the cylindrical surfaces are fixed. The problem with cylindrical surfaces free of stresses may be considered in a similar fashion.

Using the method of homogeneous solutions /1/ we obtain, as a first step, the solution of Eq.(1.1) for an infinite wedge, when

$$\tau_{\varphi z} = \begin{cases} \tau(r) & (a_1 \leqslant r \leqslant a_2,\ \varphi = \gamma) \\ 0 & (0 < r < a_1,\ a_2 < r < \infty,\ \varphi = \gamma) \end{cases}$$
$$w = 0\ (\varphi = 0)$$

In this case

$$w = w^{(1)}(r, \varphi) = \frac{1}{2G\pi i}\int_{a_1}^{a_2} \tau(\rho)\, d\rho \int_{(L)} \frac{\sin s\varphi}{s \cos s\gamma}\left(\frac{\rho}{r}\right)^s ds \tag{1.3}$$

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where the contour of integration *(L)* in the plane of the complex variable $s = \tau + i\sigma$ is a straight line parallel to the imaginary axis, provided that $-\pi/2\gamma < \tau < \pi/2\gamma$.

Then we construct the set of homogeneous solutions of Eq.(1.1) for the infinite wedge when

$$w = 0\ (\varphi = 0),\ \tau_{\varphi z} = 0\ (\varphi = \gamma)$$

and introduce their linear combination

$$w^{(2)}(r, \varphi) = \sum_{k=1}^{\infty} \sin \alpha_k \varphi \,[C_k r^{-\alpha_k} + D_k r^{\alpha_k}], \quad \alpha_k = \frac{\pi(2k-1)}{2\gamma} \tag{1.4}$$

Then the solution of the input problem is represented in the form

$$w(r, \varphi) = w^{(1)}(r, \varphi) - w^{(2)}(r, \varphi) \tag{1.5}$$

In this case the boundary conditions (1.2), except the first and the last, are satisfied. We will represent the last of conditions (1.2) in the form

$$\sum_{k=1}^{\infty} \sin \alpha_k \varphi \,[C_k R_j^{-\alpha_k} + D_k R_j^{\alpha_k}] = w^{(1)}(R_j, \varphi) \quad (j = 1, 2)$$

from which, using the orthogonality of $\sin \alpha_k \varphi$ in $[0, \gamma]$, we obtain

$$[C_k R_j^{-\alpha_k} + D_k R_j^{\alpha_k}]\frac{\gamma}{2} = \frac{\sin \alpha_k \gamma}{G\gamma\alpha_k} R_j^{-(-1)^j \alpha_k} \int_{a_1}^{a_2} \tau(\rho)\, \rho^{(-1)^j \alpha_k} d\rho \tag{1.6}$$

$$(j = 1, 2)$$

The first boundary condition of (1.2), using (1.3)-(1.5), enables us to obtain an integral equation in the unknown function of the distribution of the contact stresses

$$K_r \tau(\rho) = G\left[\delta + \sum_{k=1}^{\infty} \sin \alpha_k \gamma \,(C_k r^{-\alpha_k} + D_k r^{\alpha_k})\right] \quad (a_1 \leqslant r \leqslant a_2) \tag{1.7}$$

where the operator K_r has the form

$$K_r \tau(\rho) = \int_{a_1}^{a_2} \tau(\rho)\, k(\rho/r)\, d\rho, \quad k(y) = \frac{1}{2\pi i} \int_{(L)} \frac{\operatorname{tg} s\gamma}{s} y^s\, ds \tag{1.8}$$

Let us represent $\tau(r)$ in the form

$$\tau(r) = G\left\langle \delta\tau_0(r) + \sum_{k=1}^{\infty} [x_k^1 R_1^{\alpha_k} \tau_k^1(r) + x_k^2 R_2^{-\alpha_k} \tau_k^2(r)] \right\rangle \tag{1.9}$$

$$x_k^1 R_1^{\alpha_k} = C_k \sin \alpha_k \gamma, \quad x_k^2 R_2^{-\alpha_k} = D_k \sin \alpha_k \gamma$$

where $\tau_0(r)$, $\tau_k^j(r)$ are the respective solutions of the integral equations

$$K_r \tau_0(\rho) = 1, \quad K_r \tau_k^j(\rho) = r^{(-1)^j \alpha_k} \quad (a_1 \leqslant r \leqslant a_2,\ j = 1, 2) \tag{1.10}$$

Substituting (1.9) into (1.6), we obtain an infinite set of linear algebraic equations for determining the unknown coefficients x_k^1, x_k^2

$$x_k^1 + x_k^2 \left(\frac{R_1}{R_2}\right)^{\alpha_k} = g_k^1 + \sum_{n=1}^{\infty} [x_n^1 a_{kn}^{11} + x_n^2 a_{kn}^{12}] \tag{1.11}$$

$$x_k^1 \left(\frac{R_1}{R_2}\right)^{\alpha_k} + x_k^2 = g_k^2 + \sum_{n=1}^{\infty} [x_n^1 a_{kn}^{21} + x_n^2 a_{kn}^{22}] \quad (k = 1, 2, \ldots)$$

$$g_k^p = \frac{\delta}{\gamma\alpha_k} R_p^{-(-1)^p \alpha_k} T_k^p, \quad a_{kn}^{pj} = \frac{1}{\gamma\alpha_k} R_p^{-(-1)^p \alpha_k} R_j^{-(-1)^j \alpha_n} T_{kn}^{pj} \tag{1.12}$$

$$T_k^p = \int_{a_1}^{a_2} \tau_0(\rho)\, \rho^{(-1)^p \alpha_k} d\rho, \quad T_{kn}^{pj} = \int_{a_1}^{a_2} \tau_n^j(\rho)\, \rho^{(-1)^p \alpha_k} d\rho \tag{1.13}$$

Let us investigate the infinite system (1.11). For this we will evaluate its coefficients

$$|T_k^p| \leqslant a_p^{(-1)^p \alpha_k} T_0, \quad T_0 = \int_{a_1}^{a_2} \tau_0(\rho)\, d\rho < \infty$$

Consequently

$$|g_k^p| \leqslant \frac{\delta}{\gamma\alpha_k} \left(\frac{a_p}{R_p}\right)^{(-1)p\alpha_k} T_0 \tag{1.14}$$

To estimate the coefficients of T_{kn}^{pj}, we multiply the second of the integral equations

(1.1) by $\tau_0(r)$ and integrate within the limits from a_1 to a_2. Taking into account that for the kernel (1.8) of this equation the relation $k(y) = k(1/y)$ holds, by varying the order of integration, we obtain

$$\int_{a_1}^{a_2} \tau_k{}^j(\rho)\,d\rho = \int_{a_1}^{a_2} \tau_0(r)\, r^{(-1)^j\alpha_k} dr$$

Then

$$|T_{kn}^{pj}| \leqslant a_p^{(-1)^p\alpha_k} \left| \int_{a_1}^{a_2} \tau_n{}^j(\rho)\,d\rho \right| \leqslant a_p^{(-1)^p\alpha_k} a_j^{(-1)^j\alpha_n} T_0$$

and, consequently,

$$|a_{kn}^{pj}| \leqslant \frac{1}{\gamma\alpha_k}\left(\frac{a_p}{R_p}\right)^{(-1)^p\alpha_k}\left(\frac{a_j}{R_j}\right)^{(-1)^p\alpha_n} T_0 \tag{1.15}$$

Taking into consideration that $R_1 < a_1 < a_2 < R_2$, $\alpha_n \sim n$ $(n \to \infty)$, from the estimates (1.14) and (1.15) we conclude that the coefficients $a_{kn}{}^{pj}$ and $g_k{}^p$ of the infinite set of equations (1.11) decrease exponentially as the numbers k and n increase. Consequently, Eqs. (1.11) belong to the Poincaré-Koch type of standard systems, and their solution can be obtained using the method of reduction for any value of the problem parameters.

Thus the contact stresses under the stamp are determined by Eq. (1.9) in which $x_k{}^j$ is the solution of the infinite set of equations (1.11) and the functions $\tau_k{}^j(r)$ are solutions of the integral equations (1.10). It should be particularly noted that an exact solution can be obtained for the integral equation (1.10), which does not contain quadratures /3/.

Omitting the calculations, we will write the solution of Eqs. (1.10) in the form

$$r\tau_k{}^j(r) = \varphi_k{}^j(x), \quad r\tau_0(r) = \varphi_0(x), \quad x = \lambda \ln\frac{r}{a_1} - 1 \tag{1.16}$$

$$\varphi_k^j(x) = \lambda\varkappa_k^j\,[\varphi_k^+(x) + (-1)^j\varphi_k^-(x)], \quad \varkappa_k^j = (\sqrt{a_1a_2})^{(-1)^j\alpha_k}$$

$$\varphi_k{}^+(x) = \frac{b}{\pi\sqrt{2}}\left[\frac{C_k{}^*}{\sqrt{\operatorname{ch} b - \operatorname{ch} bx}} - \frac{\pi(2k-1)^2}{4}\eta_k(x)\right]$$

$$\varphi_k{}^-(x) = -\frac{2k-1}{2\sqrt{2}}\frac{d}{dx}\eta_k(x), \quad b = \frac{\pi}{\lambda\gamma}, \quad \lambda = 2\left(\ln\frac{a_2}{a_1}\right)^{-1}$$

$$\varphi_0(x) = b\lambda \operatorname{ch}\frac{b}{2}\left[K\left(\sqrt{1-\operatorname{th}^2\frac{b}{2}}\right)\sqrt{2(\operatorname{ch} b - \operatorname{ch} bx)}\right]^{-1}$$

$$\eta_k(x) = 2\sqrt{\operatorname{ch} b - \operatorname{ch} bx}\sum_{m=0}^{[k-1/2]}\beta_{km}\sum_{p=0}^{k-2m-1}\binom{k-2m-1}{p}\times$$

$$\frac{\operatorname{ch}^p bx\,(\operatorname{ch} b - \operatorname{ch} bx)^{k-2m-p-1}}{2(k-2m-1-p)+1}, \quad C_k{}^* = \frac{\pi\operatorname{ch}(b/2)\,P_{k-1}(\operatorname{ch} b)}{K(\sqrt{1-\operatorname{th}^2(b/2)})\,P_{-1/2}(\operatorname{ch} b)} -$$

$$\frac{\pi\operatorname{ch} b\,[P_{k-1}(\operatorname{ch} b)\,P_{-1/2}^1(\operatorname{ch} b) - P_{-1/2}(\operatorname{ch} b)\,P_{k-1}^1(\operatorname{ch} b)]}{P_{-1/2}(\operatorname{ch} b)}$$

$$\beta_{km} = (-1)^m 2^{1-k}\binom{k-1}{m}\binom{2(k-m-1)}{k-1}$$

Here $K(k)$ is the complete elliptic integral and $P_\alpha{}^n(x)$ are associated Legendre functions.

Using (1.16), we obtain expressions for the coefficients (1.11) and (1.12) of the infinite set of equations (1.11)

$$T_{kn}^{pj} = 2\varkappa_k{}^p\varkappa_n{}^j\,[T_{kn}^+ + (-1)^{j+p}T_{kn}^-], \quad T_k{}^p = 2\varkappa_k{}^pT_{k0} \tag{1.17}$$

$$T_{kn}^+ = \frac{1}{2}C_n{}^*P_{k-1}(\operatorname{ch} b) - \frac{\pi(2n-1)^2}{8}P_{k,n}$$

$$T_{kn}^- = \frac{\pi(2n-1)(2k-1)}{8}P_{k,n}, \quad T_{k0} = \frac{\pi\lambda\operatorname{ch}(b/2)\,P_{k-1}(\operatorname{ch} b)}{2K(\sqrt{1-\operatorname{th}^2(b/2)})}$$

$$P_{k,n} = \operatorname{sh} b\,[P_{n-1}(\operatorname{ch} b)\,P_{k-1}^1(\operatorname{ch} b) - P_{k-1}(\operatorname{ch} b)\,P_{n-1}^1(\operatorname{ch} b)] \times [(k-n)(k+n-1)]^{-1}, \quad k \neq n$$

$$P_{k,k} = -\operatorname{sh} b\left[\frac{d}{dk}P_{k-1}(\operatorname{ch} b)\,P_{k-1}^1(\operatorname{ch} b) - P_{k-1}(\operatorname{ch} b)\frac{d}{dk}P_{k-1}^1(\operatorname{ch} b)\right] \quad (2k-1)^{-1}$$

The formulas for the contact stresses and the coefficients of the infinite set of equations are free of quadratures, and are readily computed. Note also that to obtain acceptable practical solution of the problem for a wide range of parameter variation the infinite set of equations can be curtailed to 2-4 equations.

2. Shear by a ring sector stamp. Let us consider in cylindrical coordinates an elastic body occupying the region $R_1 \leqslant r \leqslant R_2$, $-\gamma \leqslant \varphi \leqslant \gamma$, $-\infty < z < \infty$. The cross-section

of this region for $z = \text{const}$ is called a ring sector, since the investigation of the problem considered here is based on the solution of a certain boundary value problem for a circular ring.

Problem 2 of the shear by a circular ring stamp is equivalent to the boundary value problem of the displacement function $w(r, \varphi)$ along the z-axis for Eq.(1.1) with the following boundary conditons:

$$w = \delta\ (r = R_2,\ |\varphi| \leqslant \theta),\ w = 0\ (r = R_1,\ |\varphi| \leqslant \gamma) \tag{2.1}$$
$$\tau_{rz} = G\frac{\partial w}{\partial r} = 0 \quad (r = R_2, \theta < |\varphi| < \gamma)$$
$$w = 0\ (|\varphi| = \gamma,\ R_1 < r < R_2)$$

where δ is the displacement of the stamp along the z axis, G is the shear modulus, and τ_{rz} are the tangential stresses.

The method of homogeneous solutions is also used to solve this problem /1/. As in Sect. 1, we shall first find the solution of Eq.(1.1) for the circular ring $R_1 \leqslant r \leqslant R_2$, when

$$\tau_{rz} = \begin{cases} \tau(\varphi)\ (|\varphi| \leqslant \theta,\ r = R_2) \\ 0 \quad (\theta < |\varphi| < \pi,\ r = R_2) \end{cases}, \quad w = 0\ (r = R_1) \tag{2.2}$$

In this case

$$w = w^{(1)} = \frac{1}{G}\left\langle \frac{t_0 R_2}{2} \ln\frac{r}{R_1} + \sum_{k=1}^{\infty} \frac{R_2 t_k (r^k R_1^{-k} - r^{-k} R_1^{k})}{k(\varkappa^k + \varkappa^{-k})} \cos k\varphi \right\rangle \tag{2.3}$$
$$t_k = \frac{2}{\pi}\int_0^\theta \tau(\varphi) \cos k\varphi\, d\varphi, \quad \varkappa = \frac{R_1}{R_2}$$

We then construct the set of homogeneous solutions for Eq.(1.1) for the circular ring, when

$$w = 0\ (r = R_1), \quad \tau_{rz} = 0\ (r = R_2)$$

and introduce their linear combination

$$w^{(2)} = \sum_{k=1}^{\infty} D_k W_k(r) \operatorname{ch} \beta_k\varphi, \quad W_k(r) = \left(\frac{r}{R_1}\right)^{i\beta_k} - \left(\frac{r}{R_1}\right)^{-i\beta_k} \tag{2.4}$$
$$\beta_k = \frac{\pi(2k-1)}{2\ln\varkappa}$$

It can be shown that the functions $W_k(r)$ are orthogonal in the segment $[R_1, R_2]$ with weight r^{-1}, namely, the relation

$$\int_{R_1}^{R_2} W_k(r) W_n(r) \frac{dr}{r} = \begin{cases} 0, & k \neq n \\ 2\ln\varkappa, & k = n \end{cases} \tag{2.5}$$

is satisfied.

We will represent the solution of the initial problem in the form

$$w(r, \varphi) = w^{(1)}(r, \varphi) - w^{(2)}(r, \varphi) \tag{2.6}$$

In this case the boundary conditions (2.1), except the first and the last, are satisfied. We will represent the last of conditions (2.1) in the form

$$\sum_{k=1}^{\infty} D_k W_k(r) \operatorname{ch} \beta_k \gamma = w^{(1)}(r, \gamma)$$

From this, using the orthogonality condition (2.5), we have

$$2D_k \ln\varkappa \operatorname{ch}\beta_k\gamma = \frac{R_2}{G}\left\langle -\frac{t_0 \ln\varkappa (\varkappa^{-i\beta_k} + \varkappa^{i\beta_k})}{2i\beta_k} + \frac{W_k(R_2)\operatorname{ch}\beta_k(\pi-\gamma)}{\beta_k \operatorname{sh}\pi\beta_k} \int_0^\theta \tau(\varphi)\operatorname{ch}\beta_k\varphi\, d\varphi \right\rangle \tag{2.7}$$

The first boundary condition of (2.1), with (2.3) and (2.4), enable us to obtain the integral equation for the function $\tau(\varphi)$

$$M_\varphi \tau(\psi) = \frac{t_0 \ln\varkappa}{2} + \frac{G}{R_2}\left\langle \delta + \sum_{k=1}^{\infty} D_k W_k(R_2) \operatorname{ch}\beta_k\varphi \right\rangle \quad (0 \leqslant \varphi \leqslant \theta) \tag{2.8}$$

where the operator M_φ has the form

$$M_\varphi \tau(\psi) = \frac{2}{\pi}\int_0^\theta \tau(\psi) M(\psi, \varphi)\, d\psi,$$

$$M(\psi, \varphi) = \sum_{k=1}^{\infty} \frac{\varkappa^{-k} - \varkappa^{k}}{k(\varkappa^{-k} + \varkappa^{k})} \cos k\varphi \cos k\psi \tag{2.9}$$

We represent $\tau(\varphi)$ in the form

$$\tau(\varphi) = \left(\frac{t_0 \ln \varkappa}{2} + \frac{G\delta}{R_2}\right)\left[\tau_0(\varphi) + \sum_{n=1}^{\infty} \frac{y_n \tau_n(\varphi)}{\operatorname{ch} \beta_n \gamma}\right] \tag{2.10}$$

$$D_n W_n(R_2) = \left(\delta + \frac{R_2}{2G} t_0 \ln \varkappa\right) \frac{y_n}{\operatorname{ch} \beta_n \gamma}$$

where $\tau_n(\varphi)$ are solutions of the well-known integral equations

$$M_\varphi \tau_n(\psi) = \operatorname{ch} \beta_n \varphi \quad (0 \leqslant \varphi \leqslant \theta,\ n \geqslant 0,\ \beta_0 = 0) \tag{2.11}$$

Substituting (2.10) into (2.7), we obtain the infinite set of linear algebraic equations

$$y_k = g_k + \sum_{n=1}^{\infty} a_{kn} y_n, \quad g_k = \frac{W_k^{2}(R_2) \operatorname{ch} \beta_k (\pi - \gamma)}{2 \ln \varkappa \beta_k \operatorname{sh} \pi \beta_k} T_{k,0}, \quad a_{kn} = \frac{W_k^{2}(R_2) \operatorname{ch} \beta_k (\pi - \gamma) T_{k,n}}{2\beta_k \ln \varkappa \operatorname{sh} \pi\beta_k \operatorname{ch} \beta_k \gamma} \tag{2.12}$$

$$T_{k,n} = \int_0^{\theta} \tau_n(\varphi) \operatorname{ch} \beta_k \varphi \, d\varphi \quad (k \geqslant 1, n \geqslant 0) \tag{2.13}$$

Expression (2.10) for finding the tangential stresses contains the constant

$$t_0 = \frac{T}{\pi R_2}, \quad T = R_2 \int_{-\theta}^{\theta} \tau(\varphi) \, d\varphi$$

To determine the constant T we will use the condition of statics for which we integrate Eq.(2.10) with respect to φ between the limits from $-\theta$ to θ. Having obtained the equation for T, we find

$$T = G\delta T^*, \quad T^* = P\left[1 - \frac{\ln \varkappa}{2\pi} P\right] \tag{2.14}$$

$$P = P_0 + \sum_{n=1}^{\infty} \frac{y_n}{\operatorname{ch} \beta_n \gamma} P_n, \quad P_n = \int_{-\theta}^{\theta} \tau_n(\varphi) \, d\varphi \quad (n = 0, 1, 2, \ldots)$$

Formulas (2.14) establish the relation between the stamp displacement δ and force T acting on the stamp.

Let us investigate the infinite set of equations (2.12). To do this we first evaluate the coefficients $T_{k,n}$

$$|T_{k,0}| \leqslant {}^1/_2 P_0 \operatorname{ch} \beta_k \theta, \quad |T_{k,n}| \leqslant {}^1/_2 P_n \operatorname{ch} \beta_k \theta \tag{2.15}$$

To estimate P_n we multiply the right- and left-hand sides of integral equations (2.11) by $\tau_0(\varphi)$ for $n \geqslant 1$ and integrate between the limits from $-\theta$ to θ. Taking into account the symmetry of the kernel (2.9) of Eqs.(2.11) and Eq.(2.11) when $n = 0$, and changing the order of integration, we obtain

$$P_n = \int_{-\theta}^{\theta} \tau_0(\varphi) \operatorname{ch} \beta_n \varphi d\varphi, \quad |P_n| \leqslant P_0 \operatorname{ch} \beta_n \theta \tag{2.16}$$

The estimates (2.15) and (2.16) lead to the conclusion that the coefficients g_k and a_{kn} of the infinite set of equations (2.12) decrease exponentially as the numbers k and n increase. Hence Eqs.(2.12) belong to the Poincaré-Koch type of normal systems, whose solution can be obtained using the method of reduction for any value of the problem parameters.

Taking into consideration the value of the series /4/

$$\sum_{k=1}^{\infty} k^{-1} \cos kx = -\ln\left|2 \sin \frac{x}{2}\right|$$

the integral equations (2.11) can be reduced to the form

$$\frac{2}{\pi} \int_{-\theta}^{\theta} \tau_n(\psi) k(\psi - \varphi) \, d\psi = f_n(\varphi) \quad (|\varphi| \leqslant \theta, n \geqslant 0) \tag{2.17}$$

$$f_n(\varphi) = \operatorname{ch} \beta_n \varphi$$

$$k(y) = -\ln\left|2 \sin \frac{y}{2}\right| - F(y), \quad F(y) = 2 \sum_{k=1}^{\infty} \frac{\varkappa^{2k} \cos ky}{k(1 + \varkappa^{2k})} \tag{2.18}$$

The integral equations (2.17) and (2.18) and their properties were investigated, for instance, in /5/. To find their solutions it is possible to use the method of orthogonal polynomials /5/, as the result of which we have

$$\tau_n(\varphi) = \frac{\cos(\varphi/2)}{2\sqrt{\cos\varphi - \cos\theta}} \sum_{k=0}^{\infty} a_k^n T_{2k}\left(\frac{\sin(\varphi/2)}{\sin(\theta/2)}\right) \tag{2.19}$$

where $T_{2k}(x)$ are Chebyshev polynomials, and the coefficients a_k^n are determined from the infinite sets of linear algebraic equations of the second kind, whose effective solution can be obtained using the method of reduction for any value of the parameters /5/. In the majority of cases, an acceptable practical solution can be obtained with 2-4 equations.

3. Closed solution of integral equations (2.17). We will differentiate Eq. (2.17) with respect to φ. Using the formula /4/

$$\frac{K(b)\operatorname{cn}(uK(b))}{\operatorname{sn}(uK(b))} = \frac{\pi}{2}\left[\operatorname{ctg}\frac{\pi u}{2} - 4\sum_{k=1}^{\infty}\frac{\varkappa^{2k}\sin\pi k u}{1+\varkappa^{2k}}\right],$$

$$\varkappa = \exp\left[-\frac{\pi K'(b)}{K(b)}\right]$$

where $K(b)$ is a complete elliptic integral of the first kind and $\operatorname{sn} u$, $\operatorname{cn} u$ are elliptic Jacobi functions, we obtain the integral equations /6/

$$\int_{-1}^{1} \varphi_k(\xi)\operatorname{cs}\left(\frac{\xi - x}{\eta}\right)d\xi = \frac{g_k'(x)}{K(b)} \quad (|x| \leqslant 1) \tag{3.1}$$

Here

$$\eta = \frac{\pi}{\theta K(b)}, \quad \varphi_k(\xi) = 2\tau_k(\xi\theta), \quad g_k(x) = f_k(x\theta), \quad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u}$$

Taking into account the evenness of the functions $\varphi_k(\xi)$, $g_k(x)$ we convert Eqs.(3.1) to the form

$$\int_{0}^{1} \varphi_k(\xi)\left[\operatorname{cs}\left(\frac{\xi - x}{\eta}\right) - \operatorname{cs}\left(\frac{\xi + x}{\eta}\right)\right]d\xi = \frac{g_k'(x)}{K(b)} \quad (|x| \leqslant 1) \tag{3.2}$$

It can be shown that

$$\operatorname{cs}(\xi - x) - \operatorname{cs}(\xi + x) = \frac{2\operatorname{sn} x\operatorname{cn} x\operatorname{dn}\xi}{\operatorname{sn}^2\xi - \operatorname{sn}^2 x} \tag{3.3}$$

$$\operatorname{cs}(\xi - x) + \operatorname{cs}(\xi + x) = \frac{2\operatorname{sn}\xi\operatorname{cn}\xi\operatorname{dn} x}{\operatorname{sn}^2\xi - \operatorname{sn}^2 x}$$

where $\operatorname{dn} x$ is the Jacobi elliptic function.

Using the first formula of (3.3) and, again, taking into account the evenness of the functions $\varphi_k(\xi)$, $g_k(x)$, we convert Eqs.(3.2) to the form

$$\int_{-1}^{1} \varphi_k(\xi)\operatorname{dn}\frac{\xi}{\eta}\frac{d\xi}{\operatorname{sn}(\xi/\eta) - \operatorname{sn}(x/\eta)} = \frac{g'(x)}{K(b)\operatorname{cn}(x/\eta)} \quad (|x| \leqslant 1) \tag{3.4}$$

The function $\operatorname{sn}(K\theta x/\pi)$ increases when $x \in [0,1]$, $\theta \in [0, \pi]$.

By replacing the variables using the formulas

$$\tau = \operatorname{sn}(\xi/\eta), \quad t = \operatorname{sn}(x/\eta)$$

we can reduce Eq.(3.4) to a singular integral equation of the first kind with a Cauchy kernel

$$\int_{-d}^{d} \Phi_k(\tau)\frac{d\tau}{\tau - t} = p_k(t) \quad \left(|t| \leqslant d,\ d = \operatorname{sn}\frac{1}{\eta}\right) \tag{3.5}$$

where

$$\Phi_k(\tau) = \frac{\varphi_k(\xi)}{\operatorname{cn}(\xi/\eta)}, \quad p_k(t) = \frac{\theta g_k'(x)}{\pi\operatorname{cn}(x/\eta)} \tag{3.6}$$

We will use the solution of the singular equation (3.5) in the form containing singular integrals /7/. As a result we obtain the closed solution of integral equations (2.17) in the form

$$\tau_k(\psi) = \frac{\operatorname{cn}(K\psi/\pi)}{2\pi^2\sqrt{\operatorname{sn}^2(K\theta/\pi) - \operatorname{sn}^2(K\psi/\pi)}}\Bigg[C_k - \tag{3.7}$$

$$\frac{K\theta^2}{\pi^2}\int_{-\theta}^{\theta}\frac{f_k'(\tau)\operatorname{dn}(K\tau/\pi)\sqrt{\operatorname{sn}^2(K\theta/\pi) - \operatorname{sn}^2(K\tau/\pi)}}{\operatorname{sn}(K\tau/\pi) - \operatorname{sn}(K\psi/\pi)}d\tau\Bigg]$$

where C_k are arbitrary constants. Substituting into the integral equation (2.17) for $n = 0$ the value of $\tau_0(\psi)$ defined by (3.7), taking into account that $f_0'(\tau) = 0$ and that the integral on the left is some constant for any $\varphi \in [-\theta, \theta]$, in particular for $\varphi = 0$, we have the condition for obtaining the constant C_0. From this we have

$$C_0 = \pi^3 \left\{ \int_{-\theta}^{\theta} \frac{\operatorname{cn}(K\psi/\pi)\, k(\psi)\, d\psi}{\sqrt{\operatorname{sn}^2(K\theta/\pi) - \operatorname{sn}^2(K\psi/\pi)}} \right\}^{-1} \tag{3.8}$$

where $k(\psi)$ is defined by Eq.(2.18).

The constants C_k $(k = 1, 2, \ldots)$ can be obtained using the first condition (2.16).

Note that when the functions $\varphi_k(\xi)$ and $g_k(x)$ are odd, the integral equations (2.17) can also be reduced to the singular equation (3.5) using the second of formulas (3.3).

The method of homogeneous solutions, described in Sect.2, can also be used to investigate the contact problem of the pressing of a stamp into a cylindrical surface of a ring sector, when the surfaces $|\varphi| = \gamma$ are free of tangential stresses and normal displacements.

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VIBRATION OF A CYLINDER ON AN ELASTIC LAYER PARTLY FIXED TO A RIGID BASE*

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The problem of non-resonant harmonic oscillations of an elastic cylinder on an elastic layer is considered. The contact between the cylinder and the layer is over a circular region Ω_1 of radius R_1 without friction. The layer rests on a rigid base. At the layer-base interface there are two types of contact: in the circular region Ω_2 of radius R_2 there is rigid adherence, while outside it there is no friction. The length of the projection of the distance between the centres of regions Ω_1 and Ω_2 on the horizontal plane is d. Problems of this kind are encountered in flaw detection in foundations and adhesive joints.

Problems of the vibration of a rigid body (stamp) on the surface of an elastic layer under various contact conditions were considered in /1/. Here the stamp is replaced by an elastic cylinder, which leads to a qualitatively new mechanical system that takes into account the effect of the finite elastic body. A many-sided analysis of the cylinder harmonic oscillations is given in /2/.

1. We combine the cylinder axis with the ζ axis and locate the origin of coordinates on the upper face of the layer. All quantities relating to the cylinder will be denoted by the subscript 1, and those relating to the layer by the subscript 2; λ_n, μ_n, ρ_n $(n = 1, 2)$ are the

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